ARO 25839.42-MA

2

DIFFUSION APPROXIMATIONS FOR COMPLEX REPAIR SYSTEMS

by

Donald L. Iglehart and Atam P. Lalchandani

TECHNICAL REPORT No. 66

February 1991

Prepared under the Auspices
of
U.S. Army Research Contract
DAAL-03-88-K-0063

DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

ABSTRACT

A wide variety of complex repair systems can be modeled as continuous time Markov chains. These systems are closed networks of queues with a total of n jobs circulating in the network. The process of interest is the number of jobs, $X_n(t)$, at the various repair centers at time t. After appropriate translation and scaling, we show that the processes $\{X_n(t) : t \geq 0\}$ converge weakly to a limiting multi-variate Ornstein-Uhlenbeck process. This limit process is then used to obtain computable approximations for $X_n(t)$. Numerical results are presented for three specific repairman models and the approximations are compared with exact results obtained through product form formulae. In most cases the approximation is quite accurate.

Keywords: birth/death processes, diffusion approximations, logistics, Markov chains, Ornstein-Uhlenbeck processes, repairman models, weak convergence.

DIFFUSION APPROXIMATIONS FOR COMPLEX REPAIR SYSTEMS

by

Donald L. Iglehart
Stanford University

and

Atam P. Lalchandani

1. Introduction.

For many stochastic models in applied probability complicated Markov chains arise which are impossible to analyze directly. A classical approach to this problem. dating back to BACHELIER (1900), is to show that a sequence of Markov chains with appropriate time and state scales converges at a given time point (or weakly) to a limiting diffusion process. In these instances the limiting diffusion process may hold out the only hope for providing useful approximations to practical problems.

When the Markov chains are one-dimensional birth-death processes in either discrete or continuous time, STONE (1961), (1963) has developed a complete theory for the weak convergence of these Markov chains, to a limiting diffusion. Roughly speaking, Stone's results require convergence of the infinitesimal mean and variance to those of the limiting diffusion plus convergence of boundary conditions when appropriate.

In this paper we shall apply a comparable development in higher dimensions for a restricted class of limiting diffusions: multivariate Ornstein-Uhlenbeck (m.O.U.) processes. These results will then be applied to three generalized repairman models. A special case of a m.O.U. was introduced in IGLEHART (1968) and the general case in SCHACH (1971). Problems involving the convergence of Markov chains to a m.O.U. process arise frequently in practice; see, for example, KARLIN and McGREGOR (1964), (1965), IGLEHART (1968), SCHACH (1971), and McNEIL and SCHACH (1973).

The original research was supported by the Office of Naval Research under contract N00014-72-C-0266 (NR-347-022). The revision was supported by the Army Research Office under contract DAAL03-88-K-0063.

We shall treat sequences of Markov chains in continuous time whose state spaces are subsets of Z^d, the integer lattice points of d-dimensional Euclidean space, R^d. We view elements $x \in R^d$ as column vectors. From a given point in the state space we shall only allow jumps in one step to a finite number of states. Thus multivariate birth-death processes in which transitions are only allowed to neighboring states are special cases. The typical situation for a sequence of continuous time chains, say, $\{X_n(t): t \geq 0\}$, $n = 1, 2, \ldots$, is to form a sequence of processes

$$Y_n(t) = (X_n(t) - nc)/n^{1/2},\ t \geq 0,$$

where $c \in R^d$ is a fixed vector. With this setup we would like to conclude under appropriate conditions that $Y_n(t) \Rightarrow Y(t)$, where $\Rightarrow$ denotes weak convergence and Y is a m.O.U. process. Also of interest is the convergence of the m.O.U. process as $t \to \infty: Y(t) \Rightarrow Y(+\infty)$, when this is appropriate. In applications we would approximate the random vector $X_n(t)$ by $n^{1/2}\, Y(t) + nc$ for large n.

A m.O.U. process is a d-dimensional $(d \geq 2)$ diffusion, that is a strong Markov process with continuous paths. Furthermore, if the initial state is either a constant or Gaussian, then the process is Gaussian. It is characterized by two real $d \times d$ matrices A and B, where A is symmetric and positive definite.

The *stationary transition probability density* of a m.O.U. process is given by

$$p(t,x,y) = (2\pi)^{-d/2} |\textstyle\sum(t)|^{-1/2} \exp\{-\tfrac{1}{2} f(t,x,y)\}, \tag{1.1}$$

where $x, y \in R^d$, $t > 0$,

$$f(t,x,y) = (y - \mu(t))' \textstyle\sum^{-1}(t)\ (y - \mu(t)),$$

$$\mu(t) = e^{-B\tau}x, \text{ and}$$

$$\textstyle\sum(t) = \displaystyle\int_0^t e^{-B\tau} A e^{-B'\tau} d\tau.$$

Here B' is the transpose of B. As A is symmetric, positive definite and $e^{-B\tau}$ non-singular, it is easy to show that $\sum(t)$ is symmetric, positive definite.

The convergence of a sequence of Markov processes has a long history. We mention next some of the relevant literature. KHINCHINE (1933), Chapter 3, approaches the problem through the Kolmogorov backward partial differential equation. Semi-group treatments of these problems have been given by SKOROHOD (1958), TROTTER (1958), and BURMAN (1979). The stochastic integral approach is discussed in SKOROHOD (1965), GIKHMANN and SKOROHOD (1965), and GIKHMANN (1969). An approach using martingales is developed in STROOCK and VARADHAN (1979). Still another approach can be found in BOROVKOV (1979). In the special case of birth-death processes see, in addition to the work of Stone mentioned above, the paper by LIGGETT (1979). Surveys of diffusion approximations arising in applied probability and queueing theory can be found in GLYNN (1990), and IGLEHART (1968), (1973), and (1974). For a comprehensive discussion of convergence of a sequence of Markov chains see ETHIER and KURTZ (1986).

As an example of the repairmen models that we propose to approximate by a m.O.U. process consider the following. The model consists of n operating units which are subject to stochastic failure according to an exponential failure time distribution. The operating units are backed up by m_n spare units. Failures can be of two types. With probability $p(q)$ a failure is a type 1(2) and is sent to repair facility 1(2). Repair facility 1(2) operates as a $s_n^1(s_n^2)$-server queue with exponential repair times having parameter $\mu_1(\mu_2)$. The number of units waiting for or undergoing repair at facility 1(2) is $X_n^1(t)(X_n^2(t))$. The vector $X_n(t) = (X_n^1(t), X_n^2(t))$ is a two-dimensional birth-death process with finite state space. We propose to approximate $X_n(t)$ by $n^{1/2}\,Y(t) + nc$, where c is a specific vector and $Y(t)$ is a m.O.U. process.

This paper is organized as follows. A description and properties of a m.O.U. process are given in Section 2. Convergence of continuous time Markov chains is treated in Section 3. Finally, the application of these results to three repairmen models is given in Sections 4, 5, and 6. A numerical comparison of diffusion approximations and product form solutions is given in Section 7 for several of the repairman models.

2. Multivariate Ornstein-Uhlenbeck Processes.

A d-dimensional m.O.U. process is characterized by a $d \times d$ symmetric, positive definite matrix A, a $d \times d$ matrix B, and an initial vector $Y(0)$. Let $\{W(\tau) : \tau \geq 0\}$ be a d-dimensional Brownian motion which is independent of $Y(0)$ and $A^{1/2}$ the square root of A; i.e., $A = A^{1/2}(A^{1/2})'$. Then the basic definition is the following.

Definition 2.1. *A multivariate Ornstein-Uhlenbeck process $\{Y(t) : t \geq 0\}$ is defined by the expression*

$$Y(t) = e^{-Bt}\,Y(0) + \int_0^t e^{-B(t-\tau)} A^{1/2}\,W(d\tau), \tag{2.1}$$

where the second term is the Itô stochastic integral.

Note that for $d = 1, Y(t) = e^{-B\tau}\ Y(0) + e^{-bt}\sigma \int_0^t e^{b\tau} W(d\tau)$ which is the ordinary Ornstein-Uhlenbeck process; cf. BREIMAN (1968), p. 347, COX and MILLER (1965), p. 225. In IGLEHART (1968) the special case of $B = I$ was treated, while for $B = 0$ we obtain Brownian motion with covariance at time t equal to At. From the definition of $\{Y(t): t \geq 0\}$ it is easy to show the following result.

Proposition 2.1. *The process* $\{Y(t): t \geq 0\}$ *is a continuous Markov process with stationary transition probability density given by* (1.1). *Furthermore,*

$$\mu(t) \equiv EY(t) = e^{-Bt} EY(0),$$

$$\sum(t) \equiv E[Y(t) - \mu(t)]\,[Y(t) - \mu(t)]' = \int_0^t e^{-B\tau} A e^{-B'\tau} d\tau$$

and

$$R(s, t+s) \equiv E[Y(s) - \mu(s)]\,[Y(t+s) - \mu(t+s)]' = \sum(s)\ e^{-B't}$$

If $Y(0)$ *is a constant or Gaussian, then the joint distributions of* $\{\{Y(t): t \geq 0\}$ *are Gaussian.*

Also the process given by (2.1) is a solution of the stochastic integral equation

$$Y(t) = Y(0) - B \int_0^t Y(\tau) d\tau + A^{1/2} W(t) \tag{2.2}$$

with corresponding stochastic differential equation

$$dY(t) = -BY(t) + A^{1/2} W(dt).$$

We add in passing that one could allow the matrices A and B to depend on t and define a m.O.U. process with a non-homogeneous transition function; see ARNOLD (1974), Chapter 8, for a discussion of the non-homogeneous case as well as other background material. Here, however, we prefer to keep things simple and stick with the expression (2.1).

For applications we shall be interested in the limiting behavior of $\mu(t)$ and $\sum(t)$ as $t \to \infty$. Since $\mu(t) = e^{-Bt} EY(0)$ and e^{-Bt} is non-singular, $\mu(t) \to 0$ if and only if $e^{-Bt} \to 0$. But $e^{-Bt} \to 0$ if and only if the real part of the eigenvalues of B are strictly positive; *cf.*, BROCKETT (1970), *p.* 54. Under the stated condition, the matrix equation

$$BC + CB' = A$$

has a unique solution C given by

$$C = \int_0^\infty e^{-B\tau} A \, e^{-B'\tau} d\tau = \lim_{t \to \infty} \sum(t);$$

cf. BROCKETT (1970), p. 61. It is easy to show that C is symmetric and positive definite. Also, for this case we can write $\sum(t)$ as follows:

$$\sum(t) = C - e^{-Bt} C e^{-B't}.$$

For further discussion of these problems see BELLMAN (1970), p. 239, and GANTMACHER (1959), p. 225. Summarizing, we see that if the real part of the eigenvalues of B are positive, then

$$Y(t) \Rightarrow N(0, C)$$

as $t \to \infty$, where $N(0,C)$ represents a normal random vector with mean vector 0 and covariance matrix C.

3. Convergence of Continuous-Time Markov Chains.

Let $\{X_n(t) : t \geq 0\}$ $(n = 1, 2, \dots)$ be a sequence of continuous-time Markov chains with the state space of the n^{th} chain $E_n \subset Z^d$. Denote the transition probability function of the n^{th} process by $P^{(n)}(t) = \{p_{ij}^{(n)}(t) : i, j \in E_n;\ t \geq 0\}$ and the associated Q-matrix by $Q(n) = \{q_{ij}(n) : i, j \in E_n\}$. Recall that $q_{ij}(n) = p_{ij}^{(n)\prime}(0)$. We form the sequence of processes

$$Y_n(t) = (X_n(t) - nc)/n^{1/2}, \qquad t \geq 0 \tag{3.1}$$

where the vector $c \in R^d$ is selected so that the infinitesmal mean and covariance of $Y_n(t)$ converge to those of a m.O.U. process. The vector c is called a quasi-equilibrium point and is the point to which the process $X_n(t)/n$ is attracted for large n. To find c a heuristic "mass balance" argument can be used. For $d = 1$ this amounts to balancing the upward force from the birth parameters with the downward force from the death parameters. For $d > 1$ the corresponding forces must be balanced in all d coordinate directions. Let $S_n = \{(i - nc)/\sqrt{n} : i \in E_n\}$ be the state space of $\{Y_n(t) : t \geq 0\}$.

Next define the infinitesimal mean vector per unit time for $y \in S_n$ as

$$m_n(y) = nE\{[Y_n(t+\tfrac{1}{n}) - Y_n(t)] | Y_n(t) = y\}$$

$$= n^{1/2} E\{[X_n(\tfrac{1}{n}) - X_n(0)] | Y_n(0) = y\}$$

$$= n^{1/2} \sum_{j \in E_n} (j - (nc + n^{1/2} y)) p^{(n)}_{nc+n^{1/2}y}\left(\tfrac{1}{n}\right).$$

Similarly, the infinitesimal covariance matrix per unit time is given by

$$A_n(y) = nE\left\{[Y_n(t+\tfrac{1}{n}) - Y_n(t)]\,[Y_n(t+\tfrac{1}{n}) - Y_n(t)]' | Y_n(t) = y\right\}$$

With these relations for $m_n(y)$ and $A_n(y)$ we shall assume that the following conditions hold for specified matrices A (symmetric, positive definite) and B, and some $c \in R^d$:

(3.2) the sets S_n become dense in R^d as $n \to \infty$;

(3.3) for fixed $x \in R^d$, $X_n(0) = \lfloor nc + n^{1/2} x \rfloor$ a.s.;

(3.4) there exists $J < \infty$ such that for all $n \geq 1$

$$\sup_{i,j \in E_n} \{|j - i| : q_{ij}(n) > 0\} \leq J,$$

where $|i - j|$ is the Euclidean distance between i and j;

(3.5) for all $K > 0$.

$$\lim_{n \to \infty} \sup_{\substack{y \in S_n \\ |y| < K}} |m_n(y) - By| = 0; \text{ and}$$

(3.6) for all $K > 0$,

$$\lim_{n \to \infty} \sup_{\substack{y \in S_n \\ |y| < K}} \| A_n(y) - A \| = 0$$

where for a $d \times d$ matrix D the matrix norm $\|D\| = \max_{|x|=1} |Dx|$.

Note that (3.2 and (3.3) are natural conditions only involving the state space and initial configuration of $\{Y_n(t): t \geq 0\}$. Condition (3.4) limits the size of a single jump. The most important conditions are (3.5) and (3.6) which require that the infinitesimal mean vector and covariance matrix converge uniformly in bounded subsets of S_n to the mean vector B and covariance matrix A of the limiting m.O.U. process $\{Y(t): t \geq 0\}$.

Proposition 3.1. *If $\{X_n(t): t \geq 0\}$ is a sequence of Markov chains satisfying* (3.2) - (3.6), *then for every $t \geq 0$*

$$Y_n(t) \Rightarrow Y(t)$$

as $n \to \infty$, where $\{Y(t): t \geq 0\}$ is the multivariate Ornstein-Uhlenbeck process defined in (2.1) *with $Y(0) = x$ and matrices A and B.*

The arguments in STROOCK and VARADHAN (1979), Section 11.2, can be adapted to prove this result; see PRISGROVE (1987), Theorem 2.1, for details. Also application of Rebolledo's Theorem can be used; see ETHIER and KURTZ (1986), Theorem 4.1.

4. Two Items, One Service Facility Repairman Model.

Our first model consists of two types of operating units, n_1 units of type 1 and n_2 units of type 2, where $n_1 + n_2 = n$. There are m_{n1} and m_{n2} spares for the two types of items, respectively. Both types of units are subject to failure according to independent exponential failure distributions with parameters $\lambda_1, \lambda_2 > 0$, respectively. Both types of failed units require service from a single service facility which operates like an s_n-server queue. The service times for repair at this facility are exponential with parameters $\mu_1, \mu_2 > 0$, for the two types of units, respectively. Type 1 units have preemptive priority over type 2 units for service; i.e., if on arriving, a failed unit of type 1 finds all servers busy, it preempts a type 2 unit, if any are being served. Service on type 2 units is resumed. Due to the exponential service times, the analysis is identical for the case where service is repeated. One reasonable interpretation of this model is that type 1 units are more critical than type 2 units, and the former have to be repaired as soon as possible. The flow of units is shown in Figure 1.

Let $X_n^i(t)$ denote the number of type i units waiting or undergoing repair at the service facility at time t, $i = 1,2$. The assumption of exponential failure and repair distributions means that the process $X_n(t) = (X_n^1(t), X_n^2(t))$ is a positive recurrent Markov chain with a finite state space, $E_n = \{(i,j) | 0 \leq i \leq n_1 + m_{n1},\ 0 \leq j \leq n_2 + m_{n2}\}$, as depicted in Figure 2. We shall have occasion to distinguish three regions in the state space. These are denoted by A_n, B_n and C_n in Figure 2. For all the models we shall discuss, the elements of the

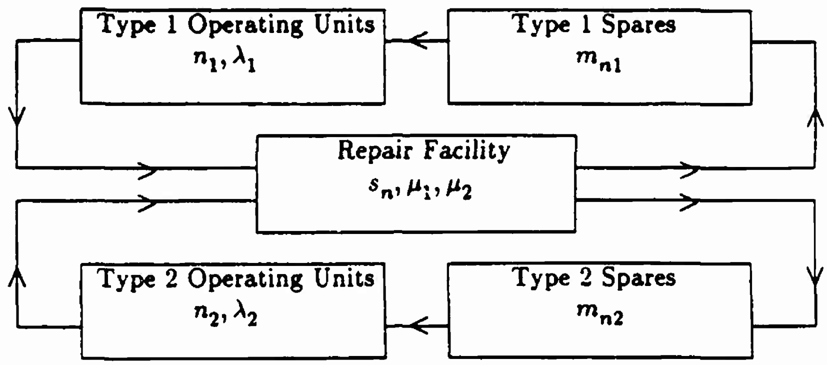

Figure 1. Two Items, One Service Facility Model

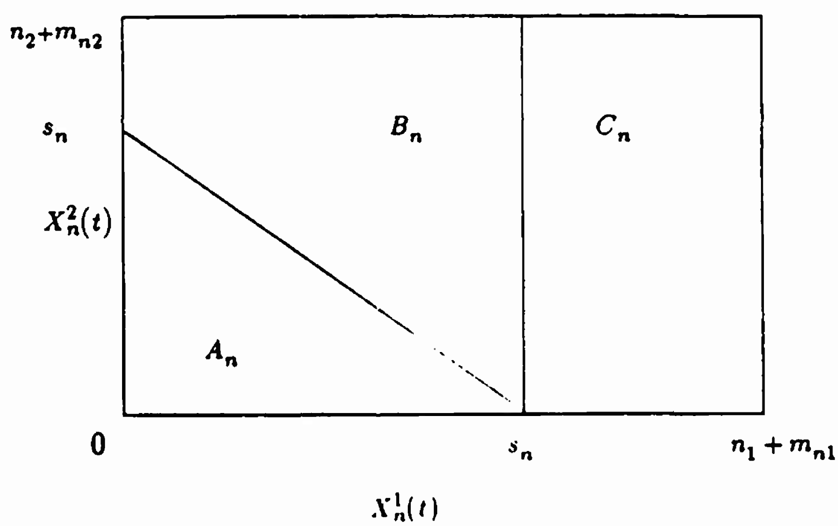

Figure 2. State Space for $X_n(t)$

matrix $Q(n) = \{q_{ij}(n) : i, j \in E_n\}$ will be denoted as follows: for $i = (i_1, i_2)$

$$q_{ij}(n) = \begin{cases} s^{(n)}(i), & j = i \\ \lambda_k^{(n)}(i), & j = i - e_k \\ \mu_k^{(n)}(i), & j = i - e_k \\ \gamma_{k\ell}^{(n)}(i), & j = i - e_k + e_\ell \\ 0, & \text{other } j, \end{cases}$$

where $k, \ell = 1, 2$, $k \neq \ell$, $s^{(n)}(i) = -\sum_{k,\ell=1}^{2} [\lambda_k^{(n)}(i) + \mu_k^{(n)}(i)] + \sum_{\substack{k,\ell=1 \\ k\neq\ell}}^{2} \gamma_{k\ell}^{(n)}(i)$, and e_k is the vector with 1 in the k^{th} position and 0 elsewhere. Table 1 outlines the infinitesimal parameters in the three regions for the process $\{X_n(t)\}$. The parameters of the system are assumed to have the following asymptotic behavior as $n \to \infty$:

$$s_n \sim ns, \qquad 0 < s < 1;$$

and

$$n_i \sim np_i, \qquad i = 1,2; \qquad p_1, p_2 > 0; \qquad p_1 + p_2 = 1;$$

$$m_{ni} \sim n_i m_i, \qquad i = 1,2; \qquad m_i > 0.$$

Parameters \ Region	A_n	B_n	C_n
$\lambda_1^{(n)}(i,j)$	$\lambda_1\{n_1 \wedge (n_1+m_{n1}-i)\}$	$\lambda_1\{n_1 \wedge (n_1+m_{n1}-i)\}$	$\lambda_1\{n_1 \wedge (n_1+m_{n1}-i)\}$
$\lambda_2^{(n)}(i,j)$	$\lambda_2\{n_2 \wedge (n_2+m_{n2}-j)\}$	$\lambda_2\{n_2 \wedge (n_2+m_{n2}-j)\}$	$\lambda_2\{n_2 \wedge (n_2+m_{n2}-j)\}$
$\mu_1^{(n)}(i,j)$	$i\mu_1$	$i\mu_1$	$s_n\mu_1$
$\mu_2^{(n)}(i,j)$	$j\mu_2$	$(s_n - i)\mu_2$	0
$\gamma_{12}^{(n)}(i,j)$	0	0	0
$\gamma_{21}^{(n)}(i,j)$	0	0	0

Table 1. Infinitesimal Parameters for the Two Item Model

This model is characterized by eight independent parameters in addition to n the total number of operating units namely, λ_1, λ_2, μ_1, μ_2, s, m_1, m_2 and p_1. In terms of these parameters we would like to be able to approximate the behavior of various processes characterizing the system, when n is large. This model satisfies the conditions spelled out in Section 3 for a sequence of continuous-time Markov processes. Thus we shall approximate $X_n(t)$ by an appropriate m.O.U. process. The approximation will depend on certain relationships among the independent parameters mentioned above. Here we shall illustrate one such case. The traffic intensity of the i^{th} item is $p_i\rho_i/s$, where $\rho_i \equiv \lambda_i/\mu_i$, for $i = 1,2$. Set $a_i = \lambda_i/(\lambda_i + \mu_i)$, $k_i = a_i p_i(1 + (\rho_i \wedge m_i))$, $\ell_i = 1$ if $\rho_i \geq m_i$ and 0 if $\rho_i < m_i$, where $i = 1,2$ for all of the constants. Suppose $k_1 + k_2 < s$. Then to solve for the vector c we find the state (i,j) in which both the left/right forces and the up/down forces are balanced. To balance the left/right forces we set

$$\lambda_1\{n_1 \wedge (n_1 + m_{n1} - i)\} - i\mu_1 = 0.$$

Divide through by n, set $i/n = c_1$, and let $n \to \infty$ to obtain

$$\rho_1\{p_1 \wedge (p_1 + p_1 m_1 - c_1)\} = c_1,$$

and solve for c_1 to find that $c_1 = k_1$. The same argument shows that $c_2 = k_2$. Next we solve for the elements of the B matrix. Select $y \in R^2$, then for n large $\lfloor nc + n^{1/2}y \rfloor \in A_n$, hence

$$m_n(y) = \begin{pmatrix} n^{-1/2}[\lambda_1\{n_1 \wedge (n_1 + m_{n1} - \lfloor nc_1 + n^{1/2}y_1 \rfloor)\} - \mu_1\lfloor nc_1 + n^{1/2}y_1 \rfloor \\ n^{-1/2}[\lambda_2\{n_2 \wedge (n_2 + m_{n2} - \lfloor nc_2 + n^{1/2}y_2 \rfloor)\} - \mu_2\lfloor nc_2 + n^{1/2}y_2 \rfloor \end{pmatrix}$$

$$= -\begin{pmatrix} \lambda_1\ell_1 + \mu_1 & 0 \\ 0 & \lambda_2\ell_2 + \mu_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + o(1),$$

where the vector $o(1)$ is uniform in y for y in a compact set of R^2. The matrix A is calculated in a similar manner. Note in this case that the eigenvalues of B are real and positive, so the $\sum(t) \to C$.

With this relationship among the parameters the vector nc lies in the region A_n for large n. If we set $X_n(0) = \lfloor nc + n^{1/2}x \rfloor$ for some $x \in R^2$ in accordance with (3.3), then for n large $X_n(0)$ will also lie in A_n. As the process $X_n(t)$ has fluctuations about nc of the order $n^{1/2}$ and the distance from the boundaries of the region A_n to the point nc is of order n, the process $X_n(t)$ never leaves the region A_n with any appreciable probability. Thus we need only be concerned with the infinitesimal mean and covariance for the process $Y_n(t)$, when $X_n(t)$ lies in A_n. Similar remarks hold for the other cases. Having obtained approximations for $X_n^i(t)$, the number of units down of type i at time t, it is easy to obtain approximations for the number of operating units of type i at time t, $Z_n^i(t) = n_i - [X_n^i(t) - m_{ni}]^+$. See Table 2 for the parameters of the m.O.U. processes.

Parameters \ Conditions	$k_1 + k_2 < s$	$k_1 + k_2 > s,\ k_1 < s$	$k_1 > s$
c	(k_1, k_2)	$\left(k_1, p_2(1+m_2) - (s-k_1)\frac{\mu_2}{\lambda_2}\right)$	$\left(p_1(1+m_1) - \frac{s\mu_1}{\lambda_1}, p_2(1+m_2)\right)$
A	$\begin{bmatrix} 2\mu_1 k_1 & 0 \\ 0 & 2\mu_2 k_2 \end{bmatrix}$	$\begin{bmatrix} 2\mu_1 k_1 & 0 \\ 0 & 2\mu_2(s-k_1) \end{bmatrix}$	$\begin{bmatrix} 2s\mu_1 & 0 \\ 0 & 0 \end{bmatrix}$
B	$\begin{bmatrix} (\lambda_1\ell_1 + \mu_1) & 0 \\ 0 & (\lambda_2\ell_2 + \mu_2) \end{bmatrix}$	$\begin{bmatrix} (\lambda_1\ell_1 + \mu_1) & 0 \\ -\mu_2 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
C	$\begin{bmatrix} \frac{k_1}{\rho_1\ell_1+1} & 0 \\ 0 & \frac{k_2}{\rho_2\ell_2+1} \end{bmatrix}$	$\begin{bmatrix} \frac{k_1}{\rho_1\ell_1+1} & \frac{\mu_2 k_1}{(\rho_1\ell_1+1)(\lambda_2+\mu_1+\lambda_1\ell_1)} \\ \frac{\mu_2 k_1}{(\rho_1\ell_1+1)(\lambda_2+\mu_1+\lambda_1\ell_1)} & \frac{s-k_1}{\rho_2} + \frac{\mu_2 k_1 \rho_2^{-1}}{(\rho_1\ell_1+1)(\lambda_2+\mu_1+\lambda_1\ell_1)} \end{bmatrix}$	$\begin{bmatrix} \frac{s}{\rho_1} & 0 \\ 0 & 0 \end{bmatrix}$

Table 2. Parameters for Limiting Process - Two Item Model

Next we make a number of qualitative remarks about the behavior of this system when n is large.

1. If the sum of the traffic intensities of the two types of items is less than 1 (light traffic), then $k_1 + k_2 < s$. The number of down units is roughly $n(c_1 + c_2) = n(k_1 + k_2) < s_n$ so no queues form. Note also that the components of $Y(+\infty)$ are independent: $c_{12} = 0$. The two item model in this case behaves exactly like two independent one item models. The fraction of type i operating units at time t, $Z_n^i(t)/n_i$, is roughly $1 - [X_n^i(t)/n_i - m_i]^+ \Rightarrow 1 - [(c_i/p_i) - m_i]^+$. Thus to insure n_i units up with high probability we only need $c_i/p_i \le m_i$. But this will be guaranteed if $\lambda_i/\mu_i \le m_i$. So to have a full complement of operating units of type i we need only provide $\lambda_i n_i/\mu_i$ spares. Any more are wasted: they just create further congestion at the repair facility.

2. If the traffic intensity of type 1 item is greater than 1 (heavy traffic), then $k_1 > s$. All but $s_n\mu_1/\lambda_1$ items of type 1 and all items of type 2 are at the service facility with high probability. All the servers are busy with type 1 units and all type 2 units simply wait in queue and are never served. In this case, it does not help to have spares of either kind in the system. Notice also that the limit process is degenerate in the second component. This case departs from our general theory in that the matrices A and C are only positive semi-definite.

3. If neither of the above two cases hold, then there is possible interaction between the two types of items. As it should be, spares of the type 2 item have no effect on the behavior of type 1 items, whereas there is a very strong dependence in the reverse case.

4. Given the independent parameters of the system, one can calculate a threshold level beyond which it does not help to add any more spares. This threshold level for the spares of item i is $n_i\lambda_i/\mu_i$ $(i = 1,2)$ for the first case mentioned above and 0 $(i = 1,2)$ for the second case. An intuitive explanation for the above result is the following: once the spares reach this threshold level, one of two cases occur—in one case, the level of units operating are at their maximum and adding more spares just adds to the pool of spares; in the other case, the service facility is congested to the point of capacity and adding more spares just adds to the congestion with no increase in the level of operating units.

5. One Item, Two Repair Facility Model.

This model consists of n units, m_n spares, and two repair facilities. The operating units are subject to failures according to an exponential failure distribution with parameter $\lambda > 0$. Two types of failures are possible. With probability $p(q)$ a failure of type one (two) occurs and the failed unit requires service from repair facility 1(2) which operates like an $s_n^1(s_n^2)$-server queue with exponential service time distribution having parameter $\mu_1(\mu_2)$. When repairs are completed on a unit, it returns to the spare pool and is once again available to be used as an operating unit. The flow of units in the system is shown in Figure 3. This is the same model considered by IGLEHART and LEMOINE (1973, 1974). Let $X_n^i(t)$ denote the number of units waiting and undergoing repair at facility $i = 1,2$. The assumption of exponential failure and repair distributions means that the process $X_n(t) = (X_n^1(t),\ X_n^2(t))$ is a positive recurrent Markov chain with a finite state space, $E_n = \{(i,j): i,j \geq 0,\ i+j \leq n+m_n\}$, depicted in Figure 4. We shall have occasion to distinguish four regions in the state space. These are labeled A_n, B_n, C_n and D_n in Figure 4. Table 3 outlines the infinitesimal parameters in the four regions for the process $X_n(t)$. As $n \to \infty$, the parameters of the system are assumed to behave as follows:

$$s_n^i \sim ns_i, \qquad 0 < s_i < 1, \qquad i = 1,2,$$

$$m_n \sim mn, \qquad m > 0\ .$$

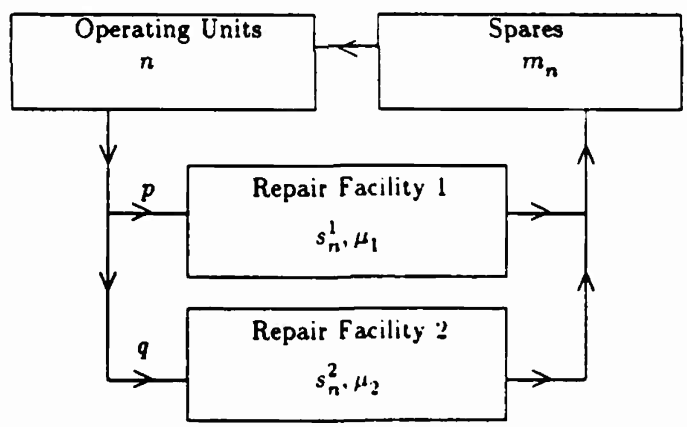

Figure 3. One Item, Two Service Facilities Model

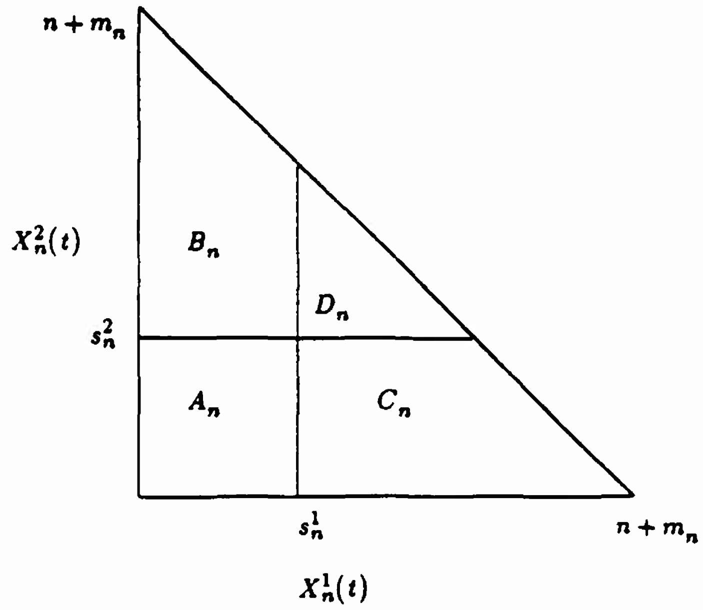

Figure 4. State Space for $X_n(t)$

Region / Parameters	A_n	B_n	C_n	D_n
$\lambda_1^{(n)}(i,j)$	$\lambda p((n+m_n-i-j)\wedge n)$	$\lambda p((n+m_n-i-j)\wedge n)$	$\lambda p((n+m_n-i-j)\wedge n$	$\lambda p((n+m_n-i-j)\wedge n)$
$\lambda_2^{(n)}(i,j)$	$\lambda q((n+m_n-i-j)\wedge n)$	$\lambda q((n+m_n-i-j)\wedge n$	$\lambda q((n+m_n-i-j)\wedge n$	$\lambda q((n+m_n-i-j)\wedge n$
$\mu_1^{(n)}(i,j)$	$i\mu_1$	$i\mu_1$	$s_n^1\mu_1$	$s_n^1\mu_1$
$\mu_2^{(n)}(i,j)$	$j\mu_2$	$s_n^2\mu_2$	$j\mu_2$	$s_n^2\mu_2$
$\gamma_{12}^{(n)}(i,j)$	0	0	0	0
$\gamma_{21}^{(n)}(i,j)$	0	0	0	0

Table 3. Infinitesimal Parameters for the Two Facilities Model

The parameters c, A, B and C of the limiting m.O.U. process have been calculated and are displayed in Table 4. The constants k_1, k_2, and ℓ are defined as follows:

$$k_2 = \frac{\lambda p}{\mu_1}\left[\frac{1+(m \wedge (\frac{\lambda p}{\mu_1}+\frac{\lambda q}{\mu_2}))}{1+\frac{\lambda p}{\mu_1}+\frac{\lambda q}{\mu_2}}\right],$$

$$k_2 = \frac{\lambda q}{\mu_2}\left[\frac{1+(m \wedge (\frac{\lambda p}{\mu_1}+\frac{\lambda q}{\mu_2}))}{1+\frac{\lambda p}{\mu_1}+\frac{\lambda q}{\mu_2}}\right],$$

and

$$\ell = \begin{cases} 1, & m < \frac{\lambda p}{\mu_1}+\frac{\lambda q}{\mu_2} \\ 0, & m \geq \frac{\lambda p}{\mu_1}+\frac{\lambda q}{\mu_2}. \end{cases}$$

Conditions \ Parameters	$s_1 > k_1,\ s_2 > k_2$	$s_1 > \frac{k_1}{k_2}\, s_2,\ s_2 < k_2$	$s_1 < k_1,\ s_2 > \frac{k_2}{k_1}\, s_1$
c	(k_1, k_2)	$\left(\frac{k_1}{k_2}, s_2,\ 1+m_2 - \frac{\mu_2 s_2}{\lambda q}\left(\frac{\lambda p}{\mu_1}+1\right)\right)$	$\left(1+m-\frac{\mu_1 s_1}{\lambda p}\left(\frac{\lambda q}{\mu_2}+1\right), \frac{k_2}{k_1}, s_1\right)$
A	$\begin{bmatrix} 2\mu_1 k_1 & 0 \\ 0 & 2\mu_2 k_2 \end{bmatrix}$	$\begin{bmatrix} \frac{2\mu_2 s_2 p}{q} & 0 \\ 0 & 2\mu_2 s_2 \end{bmatrix}$	$\begin{bmatrix} 2\mu_1 s_1 & 0 \\ 0 & 2\mu_1 s_1 \frac{q}{p} \end{bmatrix}$
B	$\begin{bmatrix} (\mu_1+\lambda p\ell) & \lambda p\ell \\ \lambda p\ell & (\mu_2+\lambda p\ell) \end{bmatrix}$	$\begin{bmatrix} (\mu_1+\lambda p) & \lambda p \\ \lambda q & \lambda q \end{bmatrix}$	$\begin{bmatrix} \lambda p & \lambda p \\ \lambda q & (\mu_2+\lambda q) \end{bmatrix}$
C	$\begin{bmatrix} \frac{k_1(\mu_1+\frac{\lambda p\ell k_2}{1+m})}{\mu_1+\lambda p\ell} & -\frac{k_1 k_2}{1+m}\ell \\ -\frac{k_1 k_2}{1+m}\ell & \frac{k_2(\mu_2+\frac{\lambda q\ell k_1}{1+m})}{\mu_2+\lambda q\ell} \end{bmatrix}$	$\begin{bmatrix} \frac{s_2\mu_2 p}{\mu_1 q} & -\frac{s_2\mu_2 p}{\mu_1 q} \\ -\frac{s_2\mu_2 p}{\mu_1 q} & \frac{s_2\mu_2\left(1+\frac{\lambda p}{\mu_1}\right)}{\lambda q} \end{bmatrix}$	$\begin{bmatrix} \frac{s_1\mu_1(1+\frac{\lambda q}{\mu_2})}{\lambda p} & -\frac{s_1\mu_1 q}{\mu_2 p}\ell \\ -\frac{s_1\mu_1 q}{\mu_2 p} & \frac{s_1\mu_1 q}{\mu_2 p} \end{bmatrix}$

Table 4. Parameters for Limiting Process - Two Facilities Model

Note that the traffic intensities at the two facilities are $\lambda p/s_1\mu_1$ and $\lambda q/s_2\mu_2$. Based on these results, the following remarks can be made regarding the behavior of the system when n is large.

1. If the traffic intensities at the two facilities are individually less than 1 (light traffic), no queues form. It does not help to increase the spares beyond $n(\lambda p/\mu_1 + \lambda q/\mu_2)$. At this critical level, n units are operating with high probability. Adding more spares only adds to the congestion at the facilities. Note that the components of $Y(+\infty)$ are independent, whenever $\ell = 0$.

2. If the traffic intensity at facility 1 is less than that at facility 2 and the latter is greater that $1 + \lambda p/\mu_1 + \lambda q/\mu_2$, the following holds true: having spares does not change the number of items at facility 1. Also, spares only increase the congestion at facility 2. So, in this case, there is no advantage in having any spares. The number of units operating is inversely proportional to the traffic intensity at facility 2. A similar statement can be made by reversing the roles of facility 1 and 2.

6. One Item, Series Facility Model.

This model consists of n units, m_n spares, and two repair facilities in series. The operating units are subject to failure according to an exponential failure distribution with parameter $\lambda > 0$. All failed units receive service at repair facility one, which operates like an s_n^1-server queue with exponential service time having parameter μ_1. With probability p, each item serviced at facility one also requires servicing at facility two, which operates like an s_n^2-server queue with exponential service time distribution having parameter μ_2. When a unit has been serviced either only at one facility or at both facilities, it returns to the spare pool and is once again available to be used as an operating unit. The flow of units in the system is shown in Figure 5.

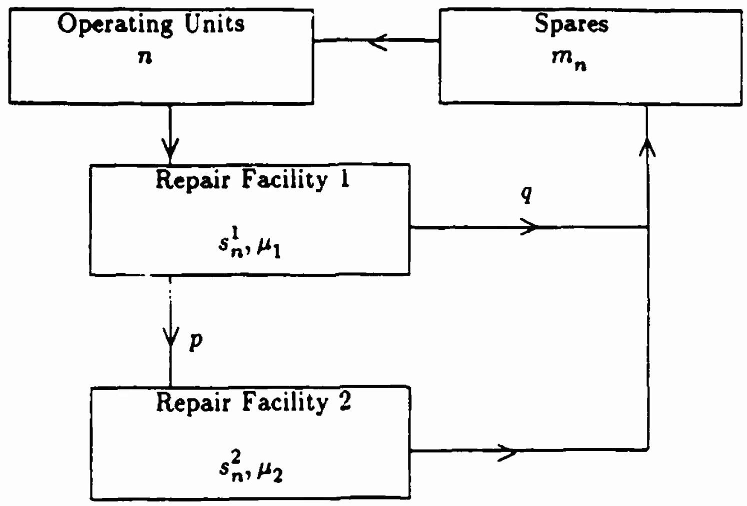

Figure 5. One Item, Series Facilities Model

Let $X_n^i(t)$ denote the number of units waiting and undergoing repair at facility $i = 1, 2$. The assumption of exponential failure and repair distributions means that the process $X_n(t) = (X_n^1(t), X_n^2(t))$ is a positive recurrent Markov chain with a finite state space $E_n = \{(i,j): i,j \geq 0,\ i + j \leq n + m_n\}$, depicted in Figure 6. We shall have occasion to distinguish four regions in the state space. These are labeled A_n, B_n, C_n, and D_n in Figure 6. Table 5 outlines the infinitesimal parameters in the four regions for the process $X_n(t)$. As $n \to \infty$, the parameters of the system are assumed to behave as follows:

$$s_n^i \sim ns_i, \qquad 0 < s_i < 1,\ i = 1,2,$$

$$m_n \sim mn, \qquad m > 0\ .$$

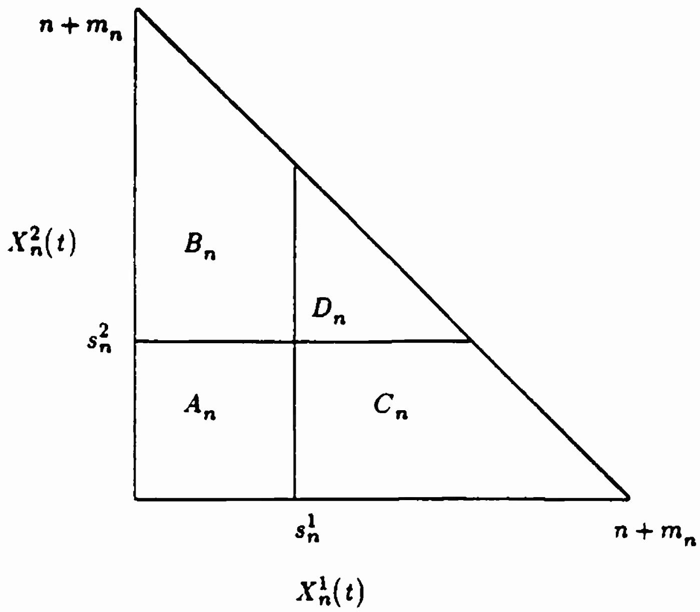

Figure 6. State Space for $X_n(t)$

Region / Parameters	A_n	B_n	C_n	D_n
$\lambda_1^{(n)}(i,j)$	$\lambda(n \wedge (n+m_n-i-j)$	$\lambda(n \wedge (n+m-i-j))$	$\lambda(n \wedge (n+m-i-j))$	$\lambda(n \wedge (n+m-i-j))$
$\lambda_2^{(n)}(i,j)$	0	0	0	0
$\mu_1^{(n)}(i,j)$	$iq\mu_1$	$iq\mu_1$	$s_n^1 q\mu_1$	$s_n^1 q\mu_1$
$\mu_2^{(n)}(i,j)$	$j\mu_2$	$s_n^2\mu_2$	$j\mu_2$	$s_n^2\mu_2$
$\gamma_{12}^{(n)}(i,j)$	$ip\mu_1$	$ip\mu_1$	$s_n^1 p\mu_1$	$s_n^1 p\mu_1$
$\gamma_{21}^{(n)}(i,j)$	0	0	0	0

Table 5. Infinitesimal Parameters for the, Series Model

Again the parameters c, A, B, and C of the limiting m.O.U. process have been calculated and are given in Table 6. The constants k_1, k_2, and ℓ are defined as follows:

$$k_1 = \frac{\lambda}{\mu_1}\left[\frac{1 + (m \wedge (\frac{\lambda}{\mu_1} + \frac{\lambda p}{\mu_2}))}{1 + \frac{\lambda}{\mu_1} + \frac{\lambda p}{\mu_2}}\right],$$

$$k_2 = \frac{\lambda p}{\mu_2}\left[\frac{1 + (m \wedge (\frac{\lambda}{\mu_1} + \frac{\lambda p}{\mu_2}))}{1 + \frac{\lambda}{\mu_1} + \frac{\lambda p}{\mu_2}}\right],$$

$$\ell = \begin{cases} 1, & m < \frac{\lambda}{\mu_1} + \frac{\lambda p}{v_2} \\ 0, & m \geq \frac{\lambda}{\mu_1} + \frac{\lambda p}{v_2}. \end{cases}$$

Here the traffic intensities at the two facilities are $\lambda/s_1\mu_1$ and $\lambda p/s_2\mu_2$. The following remarks can be made about the system for large n.

1. If the traffic intensities at the two facilities are individually less than 1 (light traffic), no queues form. It does not help to increase the spares beyond $n(\lambda/\mu_1 + \lambda p/\mu_2)$. At this critical level, n units are operating with high probability. Adding more spares only adds to the congestion at the facilities. Note that $Y^1(+\infty)$ and $Y^2(+\infty)$ are negatively correlated.

2. If the traffic intensity at facility 1 is less than that at facility 2 and the latter is greater that $1 + \lambda/\mu_1 + \lambda p/\mu_2$, then having spares does not change the number of items at facility 1. Also, spares only increase the congestion at facility 2. So, in this case, there is no advantage in having any spares. The number of units operating is inversely proportional to the traffic intensity at facility 2. A similar statement can be made by reversing the roles of facility 1 and 2.

Parameters \ Conditions	$s_1 > k_1,\ s_2 > k_2$	$s_1 > \frac{k_1}{k_2} s_2,\ s_2 < k_2$	$s_1 < k_1,\ s_2 > \frac{k_2}{k_1}$
c	$\{k_1, k_2\}$	$\left(s_2\frac{k_1}{k_2}, 1+m-s_2\frac{k_1}{k_2}(1+\frac{\mu_1}{\lambda})\right)$	$\left(1+m-s_1\frac{k_2}{k_1}\left(1+\frac{\mu_2}{\lambda p}\right), \frac{k_2}{k_1}s_1\right)$
A	$\begin{bmatrix} 2\mu_1 k_1 & -\mu_1 k_1 p \\ -\mu_1 p & 2\mu_1 k_1 p \end{bmatrix}$	$\begin{bmatrix} \frac{2\mu_2 s_2}{p} & -\mu_2 s_2 \\ -\mu_2 s_2 & 2\mu_2 s_2 \end{bmatrix}$	$\begin{bmatrix} 2\mu_1 s_1 & -\mu_1 s_1 p \\ -\mu_1 s_1 p & 2\mu_1 s_1 p \end{bmatrix}$
B	$\begin{bmatrix} (\mu_1 + \lambda\ell) & \lambda\ell \\ -\mu_1 p & \mu_2 \end{bmatrix}$	$\begin{bmatrix} (\mu_1 + \lambda) & \lambda \\ -\mu_1 p & 0 \end{bmatrix}$	$\begin{bmatrix} \lambda & \lambda \\ 0 & \mu_2 \end{bmatrix}$
C	$\begin{bmatrix} \frac{\mu_1 k_1}{\mu_1+\lambda\ell}\left[1+\frac{\lambda^2 p\ell}{\mu_1\mu_2+\lambda(\mu_1 p+\mu_2)\ell}\right] & \frac{\lambda\mu_1 k_1 p\ell}{\mu_1\mu_2+\lambda(\mu_1 p+\mu_2)\ell} \\ -\frac{\lambda\mu_1 k_1 p\ell}{\mu_1\mu_2+\lambda(\mu_1 p+\mu_2)\ell}\ell & \frac{\mu_1 k_1 p(\mu_1+\ell)}{\mu_1\mu_2+\lambda(\mu_1 p+\mu_2)\ell} \end{bmatrix}$	$\begin{bmatrix} \frac{\mu_2 s_2}{\mu_1 p} & -\frac{\mu_2 s_2}{\mu_1 p} \\ -\frac{\mu_2 s_2}{\mu_1 p} & \frac{(\lambda+\mu_1)\mu_2 s_2}{\mu_1 p} \end{bmatrix}$	$\begin{bmatrix} \frac{\mu_1 s_1}{\lambda}(\mu_2+\lambda p) & -\frac{\mu_2 s_1 p}{\mu_2} \\ -\frac{\mu_1 s_1 p}{\mu_2} & \frac{\mu_1 s_1 p}{\mu_2} \end{bmatrix}$

Table 6. Parameters for Limiting Process - Series Model

7. Numerical Example.

We note that the models described in Sections 4, 5, and 6 can be viewed as closed Jackson networks of queues. The two item, one service facility model of Section 4 requires the added complexity of different customer classes to be treated as a closed Jackson network. For that reason we do not discuss a numerical example of that model. In this section we present numerical examples of the models from Sections 5 and 6 to compare the diffusion approximation with the product form solution which is available for closed Jackson networks.

Example 1. Two Facilities Models of Section 5.

Three cases are treated here. For all cases $\mu_1 = 2$, $\mu_2 = 3$, $\lambda = 1$, $m = 0.6$. $p = 0.143$, and $q = 0.857$. The only parameters that vary are $s_1$and s_2. The three cases coincide with the three columns of Table 4. Table 7 contains the numerical values of the parameters for the limiting process for all three cases. Next we compare the approximation for the expected number of jobs at repair facility 1 and at repair facility 2 with the same values as computed using the product form solution. These comparisons were made for $n = 10$, 25, 50, and 100. For the product form calculation we used $s_n^{(i)} = \lfloor ns_i \rfloor$. The comparisons can be found in Tables 8, 9, and 10. In general the diffusion approximation is very close to the exact product form solution even when $n = 10$. The one exception occurs in Table 10 for $n = 10$, but this is a consequence of $s_1^{(n)} = \lfloor .04 \times 10 \rfloor = 0$.

Conditions / Parameters	$s_1 = 0.48 \quad s_2 = 0.64$ $s_1 > k_1, \quad s_2 > k_2$	$s_1 = 0.50 \quad s_2 = 0.20$ $s_1 > \frac{k_1}{k_2} s_2, \quad s_2 < k_2$	$s_1 = 0.04 \quad s_2 = 0.60$ $s_1 < k_2, \quad s_2 > \frac{k_2}{k_1} s_1$
c	(0.07143, 0.28571)	(0.05006, 0.84983)	(0.88075, 0.15999)
A	$\begin{bmatrix} 0.28600 & 0 \\ 0 & 1.71400 \end{bmatrix}$	$\begin{bmatrix} 0.20023 & 0 \\ 0 & 1.2 \end{bmatrix}$	$\begin{bmatrix} 0.16000 & 0 \\ 0 & 0.95888 \end{bmatrix}$
B	$\begin{bmatrix} 2.0 & 0 \\ 0 & 3.0 \end{bmatrix}$	$\begin{bmatrix} 2.14300 & 0.14300 \\ 0.85700 & 0.85700 \end{bmatrix}$	$\begin{bmatrix} 0.14300 & 0.14300 \\ 0.85700 & 3.85700 \end{bmatrix}$
C	$\begin{bmatrix} 0.07143 & 0 \\ 0 & 0.28571 \end{bmatrix}$	$\begin{bmatrix} 0.05006 & -0.05006 \\ -0.05006 & 0.75018 \end{bmatrix}$	$\begin{bmatrix} 0.71925 & -0.15981 \\ -0.15981 & 0.15981 \end{bmatrix}$

Table 7. Parameters for Two Facilities Models

n	Expected Number at Facility 1		Expected Number at Facility 2	
	Approximation	Product Form	Approximation	Product Form
10	0.715	0.707	2.857	2.870
25	1.787	1.785	7.142	7.133
50	3.575	3.575	14.280	14.280
100	7.150	7.150	28.570	28.570

Table 8. Comparison for Case $s_1 = 0.48$, $s_2 = 0.64$

n	Expected Number at Facility 1		Expected Number at Facility 2	
	Approximation	Product Form	Approximation	Product Form
10	0.501	0.494	8.498	8.284
25	1.251	1.251	21.250	21.120
50	2.503	2.503	42.490	42.370
100	5.006	5.006	84.980	84.710

Table 9. Comparison for Case $s_1 = 0.50$, $s_2 = 0.20$

n	Expected Number at Facility 1		Expected Number at Facility 2	
	Approximation	Product Form	Approximation	Product Form
10	8.807	0.000*	1.598	0.943
25	22.020	22.010	3.995	3.995
50	44.040	44.040	7.991	7.991
100	88.070	88.070	15.980	15.980

Table 10. Comparison for Case $s_1 = 0.04$, $s_2 = 0.60$

*These results are a consequence of taking $s_1^{(n)} = \lfloor .04 \times 10 \rfloor = 0$

Example 2. One Item, Series Facility Model of Section 6.

Again we treat three cases corresponding to the three columns of Table 6. For all cases $\mu_1 = 2$, $\mu_2 = 3$, $\lambda = 1$, $m = 0.6$, $p = 0.143$, and $q = 0.857$. Table 11 contains the numerical values of the parameters for the limiting process in all three cases. In Tables

12-14 we again compare the approximation and product form solution for the expected number of jobs at the two repair facilities for the case $n = 50$ and 100. The diffusion approximation is again quite close to the exact product form solution in all cases expect those in Table 13 where the error can be as large as 19%.

Parameters \ Conditions	$s_1 = 0.75$ $s_2 = 0.52381$ $s_1 > k_1$, $s_2 > k_2$	$s_1 = 0.625$ $s_2 = 0.02381$ $s_1 > \frac{k_1}{k_2} s_2$, $s_2 < k_2$	$s_1 = 0.25$ $s_2 = 0.5119$ $s_1 < k_1$, $s_2 > \frac{k_2}{k_1}$
c	$(0.5, 0.047619)$	$(0.25, 0.85)$	$(1.07669, 0.02381)$
A	$\begin{bmatrix} 2.0 & -0.143 \\ -0.143 & 0.286 \end{bmatrix}$	$\begin{bmatrix} 0.99900 & -0.07143 \\ -0.07143 & 0.14286 \end{bmatrix}$	$\begin{bmatrix} 1.0 & -0.715 \\ -0.715 & 0.143 \end{bmatrix}$
B	$\begin{bmatrix} 2.0 & 0 \\ -0.286 & 3.0 \end{bmatrix}$	$\begin{bmatrix} 3.0 & 1.0 \\ -0.286 & 0.0 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \\ 0 & 3.0 \end{bmatrix}$
C	$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.04770 \end{bmatrix}$	$\begin{bmatrix} 0.24976 & -0.24976 \\ -0.24976 & 0.74927 \end{bmatrix}$	$\begin{bmatrix} 0.52383 & -0.02383 \\ -0.02383 & 0.02383 \end{bmatrix}$

Table 11. Parameters for One Item, Series Facilities Models

n	Expected Number at Facility 1		Expected Number at Facility 2	
	Approximation	Product Form	Approximation	Product Form
50	25.0	24.61	2.383	2.346
100	50.0	49.59	4.767	4.728

Table 12. Comparison for Case $s_1 = 0.75$, $s_2 = 0.52381$

n	Expected Number at Facility 1 Approximation	 Product Form	Expected Number at Facility 2 Approximation	 Product Form
50	12.49	10.49	42.54	48.53
100	24.98	20.9[illegible]	85.08	97.06

Table 13. Comparison for Case $s_1 = 0.625$, $s_2 = 0.02381$

n	Expected Number at Facility 1 Approximation	 Product Form	Expected Number at Facility 2 Approximation	 Product Form
50	53.81	54.86	1.192	1.144
100	107.60	107.60	2.383	2.383

Table 14. Comparison for Case $s_1 = 0.25$, $s_2 = 0.5119$

While the limit theorem stated in Proposition 3.1 guarantees convergence of the sequence of approximating processes to a m.O.U. process, nothing is said about the goodness of the approximation for finite n. A method for judging how good the approximation may be was developed in PRISGROVE (1987). A numerical algorithm was constructed which computes the largest ellipsoid in R^d with center at nc within which the form of the birth and death parameters remain unchanged. The approximation to the steady-state vector has distribution which in $N(nc, nC)$. Finally, the probability that this $N(nc, nC)$ vector falls within the above ellipsoid is computed in terms of a χ_d^2 random variable. If this probability is high, we would expect a good approximation and if not we are warned to be careful about any claims made for the approximation. Numerical examples given in PRISGROVE (1987) show the usefulness of this approach.

Acknowledgement. Thanks are due to a referee who made constructive suggestions for improving the paper, Professor Tom Kurtz for providing references to the proof of Proposition 3.1, and Dr. Lindsay Prisgrove for correcting some errors in the values of the matrices A, B and C in the examples. Dr. Prisgrove also provided the numerical examples comparing the diffusion approximations and product form solutions contained in Section 7.

REFERENCES

[1] Arnold, L. (1974). *Stochastic Differential Equations: Theory and Applications.* John Wiley & Sons, New York.

[2] Bellman, R. (1970). Théorie de la spéculation. *Ann. Aci. École Norm. Sup.* 3. No. 1018, Gauthier-Villars, Paris.

[3] Billingsley, P. (1968). *Convergence of Probability Measures.* John Wiley & Sons. New York.

[4] Borovkov, A.A. (1970). Theorems on the convergence to Markov diffusion processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **16**, 47-76.

[5] Breiman, L. (1968). *Probability.*Addison-Wesley, Reading, Mass.

[6] Brockett, R.W. (1970). *Finite Dimensional Linear Systems.* John Wiley & Sons. New York.

[7] Burman, V.Y. (1979). An analytic approach to diffusion approximations in queueing. Ph.D. Dissertation. New York University.

[8] Cox, D.R. and H.D. Miller (1965). *The theory of Stochastic Processes.* John Wiley, New York

[9] Dynkin, E.B. (1965). *Markov Processes* I. Chelsea, New York.

[10] Ethier, S.N. and T.G. Kurtz (1986). *Markov Processes Characterization and Convergence.* John Wiley & Sons, New York.

[11] Gantmacher, F.R. 1959). *The Theory of Matrices*, I. Chelsea, New York.

[12] Gikhmann, I.I. (1969). On the convergence to Markov processes. *Ukrainian Math J.* 21.

[13] Gikhman, I.I. and A.V. Skorohod (1969). *Introduction to the Theory of Random Processes.*W.B. Saunders, Philadelphia. (English translation.)

[14] Iglehart, D.L. (1968). Limit theorems for the multi-urn Ehrenfest model. *Ann. Math. Statist.* **39**, 864-876.

[15] Iglehart, D.L. (1968). Diffusion approximations in applied probability. *Mathematics of the Decision Sciences*, Part 2. G.B. Dantzig and A.F. Veinott, Jr. (eds.). Amer. Math. Soc., Providence, Rhode Island.

[16] Iglehart, D.L. (1973). Weak convergence in queueing theory. *Advances in Applied Probability* **5**, 570-594.

[17] Iglehart, D.L. (1974). Weak convergence in applied probability. *Stochastic Processes Appl.* **2**, 211-242.

[18] Iglehart, D.L. and A.J. Lemoine (1973). Approximations for the repairman problem with two repair facilities, I: no spares. *Advances in Appl. Probability* 5. 595-613.

[19] Iglehart, D.L. and A.J. Lemoine (1974). Approximations for the repairman problem with two repair facilities, II: spares. *Advances in Appl. Probability* 6. 147-158.

[20] Karlin, S. and J. McGregor (1964). On some stochastic models in genetics. *Stochastic Processes in Medicine and Biology.* J. Gurland (ed.), University of Wisconsin, Madison, 245-271.

[21] Karlin, S. and J. McGregor (1965). Ehrenfest urn models, *J. Appl. Probability* 2. 352-376.

[22] Khinchine, A. (1933). *Asymptotische Gesetze der Wahrscheinlichkeitsrechung.* Springer-Verlag, Berlin.

[23] Ligget, T. (1970). On convergent diffusions: the densities and the conditioned processes. *Indiana Univ. Math. J.* **3**, 265-279.

[24] McNeil, D. and S. Schach (1973). Central limit analogues for Markov population processes. *J.R. Statist. Soc. B.* **35**, 1-23.

[25] Norman, M.F. (1972). *Markov Processes and Learning Models.* Academic Press, New York.

[26] Prisgrove, L.A. (1987). Closed queueing networks with multiple servers: transient and steady-state approximations. Ph.D. Dissertation. Department of Operations Research, Stanford University.

[27] Rosén, B. (1967a). On the central limit theorem for sums of dependent random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 7, 48-82.

[28] Rosén, B. (1967b). On asymptotic normality of sums of dependent random vectors. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 7, 95-102.

[29] Schach, S. (1971). Weak convergence results for a class of multivariate Markov processes. *Ann. Math. Statist.* **42**, 451-465.

[30] Skorohod, A. (1958). Limit theorems for Markov processes. *Theor. Probability Appl.* **3**, 202-246. (English translation.)

[31] Skorohod, A. (1965). *Studies in the Theory of Random Processes.* Addison-Wesley, Reading, Mass. (English translation.)

[32] Stone, C.J. (1961). Limit theorems for birth and death and diffusion processes. Ph.D. thesis, Stanford University.

[33] Stone, C.J. (1963). Limit theorems for random walks, birth and death processes. and diffusion processes. *Illinois J. Math.* 7, 638-660.

[34] Stroock, D.W. and S.R.S. Varadhan (1979). *Multidimensional Diffusion Processes.* Springer-verlag, Berlin.

[35] Trotter, H.F. (1958). Approximations of semi-groups of operators. *Pacific J. Math.* 8, 887-919.

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	February 1991	Technical

4. TITLE AND SUBTITLE

DIFFUSION APPROXIMATIONS FOR COMPLEX REPAIR SYSTEMS

5. FUNDING NUMBERS

DAAL03-88-K-0063

6. AUTHOR(S)

Donald L. Iglehart and Atam Lalchandani

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Department of Operations Research - 4022
Terman Engineering Center
Stanford University
Stanford, CA 94305-4022

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

ARO 25839.42-MA

11. SUPPLEMENTARY NOTES

The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

A wide variety of complex repair systems can be modeled as continuous time Markov chains. These systems are closed networks of queues with a total of n jobs circulating in the network. The process of interest is the number of jobs, $X_n(t)$, at the various repair centers at time t. After appropriate translation and scaling, we show that the processes $\{X_n(t) : t \geq 0\}$ converge weakly to a limiting multi-variate Ornstein-Uhlenbeck process. This limit process is then used to obtain computable approximations for $X_n(t)$. Numerical results are presented for three specific repairman models and the approximations are compared with exact results obtained through product form formulae. In most cases the approximation is quite accurate.

14. SUBJECT TERMS

birth/death processes, diffusion approximations, logistics, Markov chains, Ornstein-Uhlenbeck processes, repairman models, weak convergence.

15. NUMBER OF PAGES 27

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL